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Unsolvability of the Halting Problem in Quantum Dynamics

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Abstract It is shown that the halting problem cannot be solved consistently in both the Schrödinger and Heisenberg pictures of quantum dynamics. The existence of the halting machine, which is assumed from quantum theory, leads into a contradiction when we consider the case when the observer's reference frame is the system that is to be evolved in both pictures. We then show that in order to include the evolution of observer's reference frame in a physically sensible way, the Heisenberg picture with time going backwards yields a correct description.

Keywords Halting problem · Heisenberg picture · Reference frames

With the construction of universal quantum Turing machine, Deutsch proposed [1] a quantum version of the halting problem first proved by Alan Turing in 1936 [2]. In recent years, a lot of interest has been focused on quantum computation [3], and the discussion of the halting problem using a quantum computer has also received attention. Myers argued [4] that due to entanglement between a halt qubit and a system, it may be difficult to measure the halt qubit, which may spoil the computation. Subsequent discussions on the halting problem with a quantum computer have mainly focused on the superposition and entanglement of the halt qubit [5–7]. In this paper, we approach the halting scheme differently and use two pictures of quantum dynamics, i.e., the Schrödinger and Heisenberg pictures. Schrödinger's wave mechanics and Heisenberg's matrix mechanics were formulated in the early twentieth century and have been considered to be equivalent, i.e., two different ways of describing the same physical phenomenon that we observe. Therefore, in order to consider a halting scheme for a quantum system, we need to examine whether the procedure is consistent in both the Schrödinger and Heisenberg pictures. We will give an example in quantum dynamics that shows this cannot be achieved. We will then argue that it is the Heisenberg picture, rather than both pictures, that yields the correct description that not only does not run into the inconsistency shown through the halting scheme but also is physically sensible.

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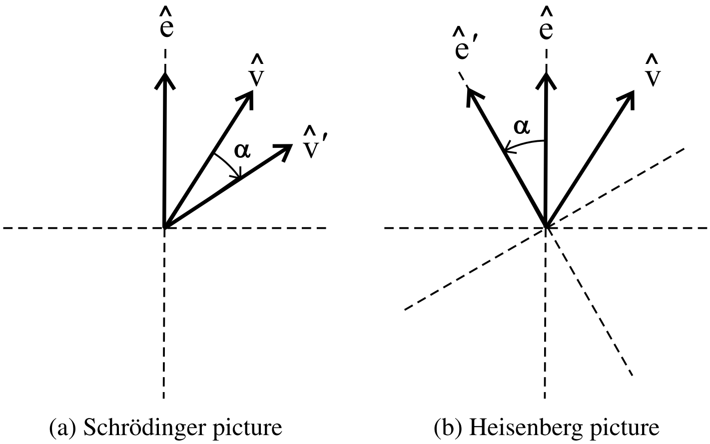


Fig. 1 For the Schrödinger picture (**a**), the unit vector $\hat{\mathbf{v}}$ evolves while the unit basis vector $\hat{\mathbf{e}}$ is intact. In the Heisenberg picture (**b**), the basis vector $\hat{\mathbf{e}}$ is rotated into opposite direction by the same amount while the unit vector $\hat{\mathbf{v}}$ remains, thereby keeping the angle between the two vectors, therefore the expectation values, the same in both pictures

In order to discuss the halting problem, we first wish to define notations to be used. In particular we will follow a similar notation used in [8, 9] such that it is convenient in both Schrödinger and Heisenberg pictures. A qubit, a basic unit of quantum information, is a two-level quantum system written as $|\psi\rangle = a|0\rangle + b|1\rangle$. Using a Bloch sphere notation, i.e., with $a = \exp(-i\phi/2)\cos(\theta/2)$ and $b = \exp(i\phi/2)\sin(\theta/2)$, a qubit in a density matrix form can be written as $|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbf{1} + \hat{\mathbf{v}} \cdot \vec{\sigma})$ where $(\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ with $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$, and $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$. Therefore a qubit, $|\psi\rangle\langle\psi|$, can be represented as a unit vector $\hat{\mathbf{v}} = (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z)$ pointing in (θ, ϕ) of a sphere with $0 \le \theta \le \pi, 0 \le \phi \le 2\pi$. A unitary transformation of a qubit in the unit vector notation $\hat{\mathbf{v}}$ can be obtained by applying U to σ_i for the corresponding ith component of the vector $\hat{\mathbf{v}}$, i.e., $\mathbf{v}_i$, where $i = x, y, z$ (also see [10] for a general transformation of a single qubit in a Bloch sphere). We will write the transformation of $\hat{\mathbf{v}}$ under the unitary operation U as $\hat{\mathbf{v}}' = U\hat{\mathbf{v}}U^\dagger$, implying the unitary transformation is applied to the corresponding σ_i. For example, let us consider the case when U is a rotation about y-axis by α in a Bloch sphere, i.e., $U = \cos\frac{\alpha}{2}|0\rangle\langle 0| - \sin\frac{\alpha}{2}|0\rangle\langle 1| + \sin\frac{\alpha}{2}|1\rangle\langle 0| + \cos\frac{\alpha}{2}|1\rangle\langle 1|$. Then it yields that $\hat{\mathbf{v}} = (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z)$ is transformed into $\hat{\mathbf{v}}' \equiv U\hat{\mathbf{v}}U^\dagger = (\cos\alpha\mathbf{v}_x + \sin\alpha\mathbf{v}_z, \mathbf{v}_y, -\sin\alpha\mathbf{v}_x + \cos\alpha\mathbf{v}_z)$. In quantum theory, there is another important variable called an observable. For a single qubit, an observable can also be written as a unit vector [9], $\hat{\mathbf{e}} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ where $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, pointing (ϑ, φ) direction in a sphere. Therefore if one is to make a measurement in (ϑ, φ) direction, the observable would be $\hat{\mathbf{e}} \cdot \vec{\sigma}$. In the Heisenberg picture of quantum theory, it is the unit basis vector $\hat{\mathbf{e}}$ that is transformed [11, p. 243]. Using a similar transformation rule as in $\hat{\mathbf{v}}$, a unitary transformation of the observable in the basis vector notation can be obtained by applying $U^\dagger$ to the σ_j by $U^\dagger\sigma_j U$ for $\mathbf{e}_j$ which we represent as $\hat{\mathbf{e}}' = U^\dagger\hat{\mathbf{e}}U$. As an example, we again consider the case when U is a rotation about y-axis by α as follows $\hat{\mathbf{e}}' \equiv U^\dagger\hat{\mathbf{e}}U = (\cos\alpha\mathbf{e}_x - \sin\alpha\mathbf{e}_z, \mathbf{e}_\mathbf{y}, \sin\alpha\mathbf{e}_x + \cos\alpha\mathbf{e}_z)$. As shown in Fig. 1, the directions of transformation for two vectors are different for Schrödinger and Heisenberg pictures. Therefore the expectation value $\hat{\mathbf{e}}' \cdot \hat{\mathbf{v}}$ in the Heisenberg picture remains the same as in the case with the Schrödinger picture, i.e., $\mathbf{e} \cdot \hat{\mathbf{v}}'$. For the remainder of this paper, we will treat the two vectors $\hat{\mathbf{v}}$ and $\hat{\mathbf{e}}$ on an equal footing. The only specialty about $\hat{\mathbf{e}}$ is that it serves as a coordinate or a basis vector such that when a measurement is made on the vector $\hat{\mathbf{v}}$, the expectation value is with respect to $\hat{\mathbf{e}}$.

Before discussing the quantum version of the halting problem, let us review the classical case [2]. Turing's halting problem asks if there is a Turing machine that performs a calculation given the description of an arbitrary Turing machine T and an input such that it is able to determine if T halts or not, indicated by an internal state h_1 or h_0, respectively. If such a machine is assumed to exist, then it is possible to construct a particular Turing machine, T_H, such that the machine does not halt, for an input T, if and only if $T(T)$ halts. However, a contradiction follows for T_H when the input is T_H itself, because $T_H(T_H)$ does not halt, if and only if, $T_H(T_H)$ halts. With a quantum system and a halt qubit replacing the internal state h_0 and h_1 in classical Turing machines, Deutsch introduced [1] a quantum version of the halting problem wherein the completion of every valid quantum algorithm through a unitary process applied to the quantum system is accompanied by the change in a halt qubit to 1 that remains 0 otherwise. We will assume such a halting machine exists and will argue that this assumption leads into a contradiction. With the introduced notations, we will consider one particular case of the halting machine, that is, when the halting machine consists of a unit vector $\hat{\mathbf{v}}_{\mathbf{s}} \equiv (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z)$ and a halt qubit $\hat{\mathbf{v}}_{\mathbf{h}} \equiv (0, 0, 1)$. We do not include an ancilla state because it will not be needed for our discussion. The time evolution of the halting machine is defined through a unitary process, and the machine halts when the unit vector $\hat{\mathbf{v}}_{\mathbf{s}}$ is rotated by δ about an arbitrary $\hat{n} = (n_x, n_y, n_z)$-axis. This time evolution of the halting machine can be achieved with the unit vector $\hat{\mathbf{v}}_{\mathbf{s}}$ evolving as follows

$$\hat{\mathbf{v}}_{\mathbf{s}} \to U_\delta \hat{\mathbf{v}}_{\mathbf{s}} U_\delta^\dagger \tag{1}$$

where $U_\delta \equiv \cos(\delta/2)\mathbf{1} - i\sin(\delta/2)(\hat{n} \cdot \vec{\sigma})$ and the halt qubit $\hat{\mathbf{v}}_{\mathbf{h}}$ is transformed into $-\hat{\mathbf{v}}_{\mathbf{h}}$ with a unitary operation σ_x. In the following, we will show this halting machine runs into a contradiction.

Before we proceed with our discussion of the halting problem, we wish to discuss the concept of observables in quantum theory. When we want to check a moving vehicle's speed, we may use a speed gun and could read, for example, 80 km/hr. Or we could use a thermometer to measure a room temperature which may yield, for example, 25 degrees Celsius. While the measurement tools, such as the speed gun and the thermometer, yield the output with not only numbers but also units such as km/hr and degrees Celsius, what the actual measurement yields is rather different. For example, a laser speed gun checks the distances from the gun at two different times and is designed to calculate and to yield an output of the moving vehicle's speed. A mercury-thermometer is designed to show the temperature in relation to the increase of the volume of mercury in the thermometer. The numbers obtained from the measurement represent the perception experienced by an observer and the meaning of those numbers, such as speed or temperature represented with units, a concept, is imposed by an observer. In quantum theory, concepts such as position and momentum are called observables and the numbers that result from the measurements are represented as eigenvalues [11, p. 63].

Let us take an example of a one-dimensional line as shown in Fig. 2. In order to claim a dot, which is lying on the line, is either on the right or on the left, there should be a reference point. For example, with respect to the origin or with respect to $+3$, one may say the dot is on the left or on the right. Instead of looking at the line from outside, suppose there is an observer being confined to the one-dimensional line facing into the paper as shown in Fig. 2c. The observer measures or perceives whether the dot is on the right or on the left. Depending on where the observer is sitting, the outcome of the measurement, i.e., either on

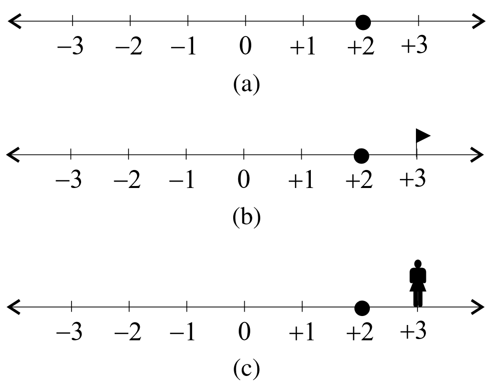


Fig. 2 For (**a**), it is not possible to claim the black dot is on the right or on the left. In (**b**), we may say, with respect to the flag in +3, the dot is on the left. If we assume there is an observer living and sitting at +3 while facing into the paper (i.e., the same direction as the reader of this paper) as in (**c**) and if the observer measures and obtains the result that the dot is on the left, then it is the observer who is serving the role of the flag in (**b**), i.e., as a reference point

the right or on the left, will change. In this case, we note that the observer him or herself is serving the role of the reference point. Therefore when the observer makes a measurement and gets a result that the dot is on the right or on the left, this implies that with respect to his or her reference frame of the position on the line, the dot is on the right or on the left. Let us apply the same logic to the case of a single qubit in a Bloch sphere. When an observer measures a qubit in a certain direction, say in $\hat{n}$, the outcome of the measurement is either $+1$ or -1. The eigenvalue obtained is with respect to the measurement direction $\hat{n}$. It is noted that $\hat{n}$ is playing a similar role as the reference point in the case of the one-dimensional line example. We also note that the measurement outcome of $+1$ or -1 is the perception experienced by the observer. That is, it is the observer who obtains the outcome $+1$ or -1. Therefore, the outcome should be meaningful with respect to the observer's certain reference frame. Because we already know that the eigenvalue outcome $+1$ or -1 is meaningful with respect to the measurement direction $\hat{n}$, it leads us to consider the observer's reference frame as $\hat{n}$ for our single qubit measurement case. Using the unit vector notations we previously defined, we propose the following:

Postulate 1 *Given a unit vector* $\hat{\mathbf{v}}$, *an observer's reference frame is identified with a basis unit vector* $\hat{\mathbf{e}}$.

With this postulate, two pictures of quantum theory can have a natural physical realization between an observer and a system. Figure 1 shows that, in the Schrödinger picture, the observer's reference frame, represented by the unit basis vector $\hat{\mathbf{e}}$, stays still while the state vector is rotated clockwise by α, and the Heisenberg picture shows the unit vector stays still and the observer's coordinate is rotated counterclockwise by α. In both cases, the observer would observe exactly the same phenomenon.

It should be noted that we are not using a notion of detector or apparatus in the place of an observer. According to our postulate, for a given unit vector, the observer's reference frame is represented with a unit basis vector in a Bloch sphere. However, it was shown that [12] a finite dimensional detector cannot encode an arbitrary unitary transformation whereas, according to our postulate, the observer's identified coordinate unit basis vector represents an arbitrary measurement basis for a given qubit. Therefore, we do not use the term detector or an apparatus to replace an observer. If one wants to include an apparatus or detector, we may consider the state, i.e., $\hat{\mathbf{v}}$, to be a larger system that includes a qubit and an apparatus and the coordinate vector for an observer would also be represented by the same larger basis vector. However, in this paper, we only consider the simplest possible case of a single qubit.

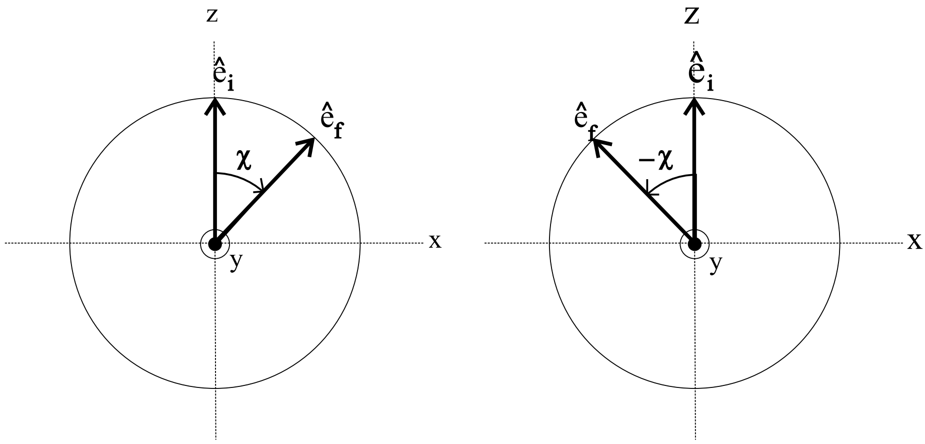
Let us now consider a system with an observer and the halting machine defined with the evolution in (1). That is, we are considering a closed system consisted of a quantum state, represented by the unit vector $\hat{\mathbf{v}}_{\mathbf{s}} = (\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z)$, an observer, whom we call Alice, represented by the reference frame $\hat{\mathbf{e}}_{\mathbf{s}} = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ introduced above, and a halt qubit $\hat{\mathbf{v}}_{\mathbf{h}}$ along with Alice's reference frame for the halt qubit defined as $\hat{\mathbf{e}}_{\mathbf{h}} \equiv (0, 0, 1)$. Alice is to transform the unit vector $\hat{\mathbf{v}}_{\mathbf{s}}$ by δ about an arbitrary $\hat{n} = (n_x, n_y, n_z)$-axis with $U_\delta = \cos(\delta/2)\mathbf{1} - i\sin(\delta/2)(\hat{n}\cdot\vec{\sigma})$ and also applies σ_x on a halt qubit such that $\hat{\mathbf{v}}_{\mathbf{h}} \to -\hat{\mathbf{v}}_{\mathbf{h}}$. If Alice were to measure the evolved vector state, the expectation value would be $\hat{\mathbf{e}}_{\mathbf{s}} \cdot (U_\delta \hat{\mathbf{v}}_{\mathbf{s}} U_\delta^\dagger)$. Next, we wish to consider the same procedure in the Heisenberg picture. In the Schrödinger picture we discussed above, the unitary evolution was performed on $\hat{\mathbf{v}}_{\mathbf{s}}$. Therefore, in the Heisenberg picture, the $U_\delta^\dagger$ transforms the basis vector $\hat{\mathbf{e}}_{\mathbf{s}}$ into $U_\delta^\dagger \hat{\mathbf{e}}_{\mathbf{s}} U_\delta$ where $U_\delta^\dagger = \cos(\delta/2)\mathbf{1} + i\sin(\delta/2)(\hat{n}\cdot\vec{\sigma})$ and the observable for the halt qubit, i.e., $\hat{\mathbf{e}}_{\mathbf{h}}$, is transformed into $-\hat{\mathbf{e}}_{\mathbf{h}}$. It yields the expectation value of $(U_\delta^\dagger \hat{\mathbf{e}}_{\mathbf{s}} U_\delta) \cdot \hat{\mathbf{v}}_{\mathbf{s}}$ which is equal to the expectation value in the Schrödinger picture, $\hat{\mathbf{e}}_{\mathbf{s}} \cdot (U_\delta \hat{\mathbf{v}}_{\mathbf{s}} U_\delta^\dagger)$.

We now consider the halting machine in (1) with one particular input. That is, when the input state to be transformed is the Alice's unit basis vector, i.e., $\hat{\mathbf{v}}_{\mathbf{s}} = \hat{\mathbf{e}}_{\mathbf{s}}$. Note that we are treating $\hat{\mathbf{v}}$ and $\hat{\mathbf{e}}$ on an equal footing. In the Schrödinger picture, the evolution is, $\hat{\mathbf{e}}_{\mathbf{s}} \to U_\delta \hat{\mathbf{e}}_{\mathbf{s}} U_\delta^\dagger \equiv \hat{\mathbf{e}}_{\mathbf{s}}''$, and Alice also transforms $\hat{\mathbf{v}}_{\mathbf{h}} \to -\hat{\mathbf{v}}_{\mathbf{h}}$. We now consider the same procedure in the Heisenberg picture. In this case, the unit basis vector $\hat{\mathbf{e}}_{\mathbf{s}}$, is transformed as $\hat{\mathbf{e}}_{\mathbf{s}} \to U_\delta^\dagger \hat{\mathbf{e}}_{\mathbf{s}} U_\delta \equiv \hat{\mathbf{e}}_{\mathbf{s}}'''$ and $\hat{\mathbf{e}}_{\mathbf{h}} \to -\hat{\mathbf{e}}_{\mathbf{h}}$. Note that $\hat{\mathbf{e}}_{\mathbf{s}}'' \neq \hat{\mathbf{e}}_{\mathbf{s}}'''$ unless $\hat{\mathbf{e}}_{\mathbf{s}} = \pm\hat{n}$ or $\delta = k\pi$ where $k = 0, 1, 2, \ldots$. For the example of a system with an observer and the halting machine studied in the previous paragraph, the vector $\hat{\mathbf{v}}_{\mathbf{s}}$ has evolved, with respect to $\hat{\mathbf{e}}_{\mathbf{s}}$, into the same output in both Schrödinger and Heisenberg pictures. Similarly, with respect to $\hat{\mathbf{e}}_{\mathbf{h}}$, the halt qubit, $\hat{\mathbf{v}}_{\mathbf{h}}$, halted in both pictures. However, in the case with $\hat{\mathbf{e}}_{\mathbf{s}}$ as an input we just considered, while the halt qubit $\hat{\mathbf{v}}_{\mathbf{h}}$ halted on both occasions with respect to $\hat{\mathbf{e}}_{\mathbf{h}}$, the vector that is being evolved, i.e., $\hat{\mathbf{e}}_{\mathbf{s}}$, turned out as two generally different outputs in two pictures. This contradicts our assumption about the halting machine in (1) because the machine should yield an output that is a rotation of the input by δ about an $\hat{n}$-axis and is unique.

Therefore, we have shown that the existence of the halting machine that is assumed from quantum theory leads into a contradiction when we consider the input of the unit basis vector $\hat{\mathbf{e}}_{\mathbf{s}}$ (for simplicity, we will omit the subscript **s** from now on), which is transformed into two generally different outputs in Schrödinger and Heisenberg pictures. However, not only can the halting problem not be solved consistently in both pictures, but also the evolution of the unit basis vector $\hat{\mathbf{e}}$ is physically sensible in neither of the two pictures in quantum dynamics. With our first postulate, we were able to impose a physical meaning on the Schrödinger and Heisenberg pictures of quantum theory. That is, in case of the Schrödinger picture, the system is evolving while an observer's reference frame is intact and, for the Heisenberg's picture, an observer's coordinate is evolving and the system is staying still. The equivalence of these two pictures comes from the fact that the observer would observe the same phenomenon and would not be able to tell the difference between them. For example, an observer applying a unitary operation to a qubit is experiencing a unitary evolution being applied to the qubit and this experience is the same in both pictures. But when it is the observer's reference frame that is evolving, it is difficult to imagine how an observer could observe or experience it. As shown in Fig. 3a, let us assume that initially vector $\hat{\mathbf{e}}$ is pointing z-direction and with the unitary operation of rotation about y-axis, $\hat{\mathbf{e}}$ evolves under

$$U = e^{-i\sigma_y t/2} \tag{2}$$

in the Schrödinger picture. And the final state of $\hat{\mathbf{e}}$ would be rotated by χ after time t, which we write as $\chi(t)$. The difficulty with this evolution is that in order to experience the unitary



(a) Schrödinger picture (b) Heisenberg picture

Fig. 3 Unitary evolution of $\hat{\mathbf{e}}$ is considered. The vector $\hat{\mathbf{e}}$ is initially pointing z-direction and is rotated about y-axis by χ after time t. In the Schrödinger picture as in (**a**), the vector is rotated clockwise and in (**b**) the vector is rotated by $-\chi$

evolution, Alice needs to be in another reference frame, say $\chi'(t)$. However, $\hat{\mathbf{e}}$ itself is Alice's reference frame and there cannot be another reference frame. Similarly, in the Heisenberg picture, $\hat{\mathbf{e}}$ evolves under

$$U^\dagger = e^{i\sigma_y t/2} \tag{3}$$

As shown in Fig. 3, the vector is being rotated counterclockwise and is in $-\chi(t)$. In this case, for the observer in the reference frame of $-\chi(t)$, there needs to be additional vector in $\chi'(t)$ in order for Alice to experience the evolution of $\hat{\mathbf{e}}$. Again, this is not possible because $-\chi(t)$ is not only Alice's reference frame but also the system vector. Therefore, in order to have a satisfactory picture of Alice observing her own reference frame's evolution, Alice needs another reference frame or another vector.

Therefore, it is not possible for either picture to be the correct way to describe the observer's experience of the evolution of $\hat{\mathbf{e}}$. Because $\hat{\mathbf{e}}$ is serving the role of both what the observer experiences and the observer's own reference frame, we need a picture such that the evolution of $\hat{\mathbf{e}}$ is neither of them yet yields the observer's experience of $\hat{\mathbf{e}}$'s evolution. In order to resolve the dilemma discussed above and to determine the correct description for the observer's experience, we introduce our second postulate as follows:

Postulate 2 *What an observer observes or experiences must be time forwarding.*

Note that we are only postulating that the observer's experience is time forwarding and not necessarily the whole system, i.e., including the physical system and the observer, is time forwarding.

Let us re-consider the evolution of $\hat{\mathbf{e}}$ under the Heisenberg picture. Note that for the unitary operation in (3), it is possible to change the signs of t and σ_y while keeping the whole unitary operator the same, that is

$$U^\dagger = e^{-i\sigma_y(-t)/2} \tag{4}$$

This corresponds to the vector evolving to χ while t is going to the minus direction compared to the previous Heisenberg case wherein the vector evolved to $-\chi$ with time going forward. In this case, we note that the observer cannot be in the reference frame $\chi(-t)$ because from the second postulate, we assumed what the observer observes or experiences is only time forwarding. If Alice is in the reference frame that is moving backward in time, she would observe everything going backward in time. However, from the assumption we made with the second postulate, this is not possible. We may consider the same trick with Schrödinger picture evolution, that is, by putting minus signs for both time and σ_y. But in this case, it still requires an additional observer's reference frame because the observer who is in the reference frame with time forwarding would simply observe χ in $+t$. This is similar to the way an electron in the negative energy would appear as a positron in the positive energy to an observer who is also in the positive energy. Therefore, in the Schrödinger picture, this new view still requires an additional reference frame and is not satisfactory.

Therefore, with two postulates, in order to have a satisfactory description of experiencing the evolution of $\hat{\mathbf{e}}$ as well as of $\hat{\mathbf{v}}$, we are forced to conclude that the quantum evolution follows according to the Heisenberg picture, not the Schrödinger picture, with time going backwards as shown in (4). Moreover, it leads us to abandon the general picture having the observer being in a certain reference frame evolving in time and observing the other vector. In other words, the more familiar picture of the observer being in the reference frame that is evolving forward in time should be abandoned, and the observer should be identified as what is being observed, i.e., χ, and its association with time, t.

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